Full Ext-exceptional collections for acyclic quivers

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Decompositions of triangulated categories

Let $\mathcal{D}$ be a $\mathbb{C}$-linear triangulated category of finite type.

$\exists$ "decompositions" of the triangulated category $\mathcal{D}$.

- A heart (of a bounded t-structure).
- A full (Ext-)exceptional collection.

Macrì constructed a correspondence

$$\begin{array}{ccc} \{\text{full Ext-exceptional collection}\} & \longrightarrow & \{\text{heart of a bounded } t\text{-structure}\} \\ \mathcal{E} & \mapsto & \langle \mathcal{E} \rangle_{\mathrm{ex}} \end{array}$$

Here $\langle \mathcal{E} \rangle_{\mathrm{ex}}$ is the extension closure of $\mathcal{E}$.

Question

When can we obtain a heart from a full Ext-exceptional collection?

Main result

Let Q be an acyclic quiver and $\mathcal{D} = \mathcal{D}^b(Q) := \mathcal{D}^b\mathrm{mod}(\mathbb{C}Q)$.

Consider the following two conditions:

(A1) For each $i, j \in Q_0$, we have $\#\{i \to j \in Q_1\} \leq 1$.

(A2) Let $i, j, k \in Q_0$ such that $i < j < k$. If there are arrows from i to j and from j to k, then there are no arrows from i to k.

Theorem 1 (O).

Let Q be an acyclic quiver satisfying (A1) (A2), and $\mathcal{A}$ a heart in $\mathcal{D}^b(Q)$. Assume that $\mathcal{A}$ is obtained from $\mathrm{mod}(\mathbb{C}Q)$ by iterated simple tilts. Then, there exists a full Ext-exceptional collection $\mathcal{E} = (E_1, \ldots, E_\mu)$ such that $\mathcal{A} = \langle \mathcal{E} \rangle_{\mathrm{ex}}$ and $\mathrm{Sim}\,\mathcal{A} = \{E_1, \ldots, E_\mu\}$.

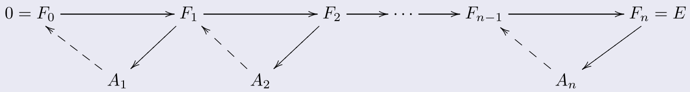
A heart of a bounded t-structure

Let $\mathcal{D}$ be a $\mathbb{C}$-linear triangulated category of finite type.

Definition 2.

A full additive subcategory $\mathcal{A} \subset \mathcal{D}$ is a heart (of a bounded t-structure) in $\mathcal{D}$ if it satisfies the following conditions:

- *For any $E, F \in \mathcal{A}$, we have $\mathrm{Hom}_{\mathcal{D}}(E, F[p]) \cong 0$ for $p < 0$.*
- *For any nonzero object $E \in \mathcal{D}$, there exists a sequence of integers $k_1 > k_2 > \cdots > k_n$ and a sequence of exact triangles*



where $A_i \in \mathcal{A}[k_i]$ for all $i = 1, \ldots, n$.

It is known that a heart $\mathcal{A}$ in $\mathcal{D}$ has a structure of an abelian category.

Remark: A heart is usually defined as the “center” of a (bounded) t-structure. In this talk, we use an equivalent definition for simplicity.

A heart $\mathcal{A}$ is said to be of finite length if it is Artinian and Noetherian.
This is equivalent to the following property:
For any nonzero object $E \in \mathcal{A}$, there exists a filtration

$$0 = F_0 \subset F_1 \subset \cdots \subset F_m = E$$

such that F_i/F_{i-1} is simple in $\mathcal{A}$.

Example:
Let $\mathcal{A}$ be an abelian category $\mathcal{A}$. The category $\mathcal{A}$ forms a heart of finite length in its derived category $\mathcal{D}^b(\mathcal{A})$. This is called the standard heart.

Therefore, a derived category of an Abelian category admits a heart.

Simple tiltings

Let $\mathcal{A}$ be a heart in $\mathcal{D}$.
Denote by $\operatorname{Sim}\mathcal{A}$ the set of isomorphism classes of simple objects in $\mathcal{A}$.

For $S \in \operatorname{Sim}\mathcal{A}$, define full subcategories ${}^{\perp}S$ and $S^{\perp}$ by

$$ {}^{\perp}S := \{E \in \mathcal{A} \mid \operatorname{Hom}_{\mathcal{A}}(E, S) = 0\}, \quad S^{\perp} := \{E \in \mathcal{A} \mid \operatorname{Hom}_{\mathcal{A}}(S, E) = 0\}. $$

Then, we have $\mathcal{A} = \langle {}^{\perp}S, S \rangle_{\mathrm{ex}} = \langle S, S^{\perp} \rangle_{\mathrm{ex}}$.

Define $\mathcal{A}_S^{\sharp}$ and $\mathcal{A}_S^{\flat}$ by

$$ \mathcal{A}_S^{\sharp} := \langle S[1], {}^{\perp}S \rangle_{\mathrm{ex}}, \quad \mathcal{A}_S^{\flat} := \langle S^{\perp}, S[-1] \rangle_{\mathrm{ex}}. $$

It is known that $\mathcal{A}_S^{\sharp}$ and $\mathcal{A}_S^{\flat}$ are hearts in $\mathcal{D}$. The heart $\mathcal{A}_S^{\sharp}$ (resp., $\mathcal{A}_S^{\flat}$) is called the forward simple tilt (resp., backward simple tilt).

Let Q be an acyclic quiver and $\mathcal{A}$ a heart in $\mathcal{D} = \mathcal{D}^b(Q)$.

It is known that if $\mathcal{A}$ is of finite length then so are $\mathcal{A}_S^\sharp$ and $\mathcal{A}_S^\flat$.
Moreover, if $\mathcal{A}$ has finitely many simple objects then so are $\mathcal{A}_S^\sharp$ and $\mathcal{A}_S^\flat$.

In particular, in the cases of Dynkin quivers the following result is known. A Dynkin quiver $\vec{\Delta}$ is an (acyclic) quiver whose underlying graph is one of the Dynkin diagrams.

Proposition 3 (Keller–Vossieck).

Let $\vec{\Delta}$ be a Dynkin quiver.
Any heart in $\mathcal{D}^b(\vec{\Delta})$ can be obtained from $\mathrm{mod}(\mathbb{C}\vec{\Delta})$ by iterated simple tilts.

Definition

Let $\mathcal{D}$ be a $\mathbb{C}$-linear triangulated category of finite type.

Definition 4.

1. *A object $E \in \mathcal{D}$ is exceptional if*

$$\mathrm{Hom}_{\mathcal{D}}(E, E[p]) \cong \begin{cases} \mathbb{C}, & p = 0, \\ 0, & p \neq 0. \end{cases}$$

2. *An ordered set $\mathcal{E} = (E_1, \dots, E_\mu)$ is called exceptional collection if*
   - *$E_1, \dots, E_\mu$ are exceptional,*
   - *For $i > j$, we have $\mathrm{Hom}_{\mathcal{D}}(E_i, E_j[p]) \cong 0$ for any $p \in \mathbb{Z}$.*
3. *An exceptional collection $\mathcal{E}$ is full if the smallest full triangulated subcategory of $\mathcal{D}$ containing $\mathcal{E}$ is equivalent to $\mathcal{D}$.*

Remark: The existence of a full exceptional collection is stronger than the existence of a bonded t-structure.

Recall that in the derived category $\mathcal{D}^b(\mathcal{A})$ of an Abelian category $\mathcal{A}$ we have

$$\operatorname{Ext}^p_{\mathcal{A}}(E,F) \cong \operatorname{Hom}_{\mathcal{D}^b(\mathcal{A})}(E,F[p]), \quad p \in \mathbb{Z},\ E,F \in \mathcal{A}.$$

Definition 5 (Macrì).

A full exceptional collection $\mathcal{E} = (E_1, \ldots, E_\mu)$ is Ext if for any i, j we have

$$\operatorname{Hom}_{\mathcal{D}}(E_i, E_j[p]) \cong 0 \quad \text{for} \quad p \leq 0.$$

If an object E is exceptional, so is its shift $E[p]$ for some $p \in \mathbb{Z}$.

Hence, for any full exceptional collection $(E_1, \ldots, E_\mu)$, one can choose $(p_1, \ldots, p_\mu) \in \mathbb{Z}^\mu$ so that $(E_1[p_1], \ldots, E_\mu[p_\mu])$ is full Ext-exceptional collection.

Example

Let $Q = (Q_0, Q_1)$ be an acyclic quiver and $Q_0 = \{1, \ldots, \mu\}$.
Assume that if $i > j$ then there are no arrows from $i \in Q_0$ to $j \in Q_0$.

Denote by S_i the simple $\mathbb{C}Q$-module with respect to the vertex $i \in Q_0$.
The $\mathbb{C}Q$-modules $S_1, \ldots, S_\mu$ are exceptional.

For $i \neq j$, we have

$$\begin{aligned} \dim_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}Q}(S_i, S_j) &= 0, \\ \dim_{\mathbb{C}} \mathrm{Ext}^1_{\mathbb{C}Q}(S_i, S_j) &= \#\{i \to j \in Q_1\}. \end{aligned}$$

and $(S_1, \ldots, S_\mu)$ forms a full Ext-exceptional collection in $\mathcal{D}^b(Q)$.

Known result

Proposition 6 (Macrì).

Let $\mathcal{E} = (E_1, \dots, E_\mu)$ be a full Ext-exceptional collection.
The extension closure $\langle \mathcal{E} \rangle_{\mathrm{ex}} \subset \mathcal{D}$ is a heart of finite length such that $\mathrm{Sim}\, \langle \mathcal{E} \rangle_{\mathrm{ex}} = \{E_1, \dots, E_\mu\}$.

$$\begin{array}{ccc} \{\text{full Ext-exc. coll.}\} & \longrightarrow & \{\text{heart of finite length}\} \\ \mathcal{E} & \mapsto & \langle \mathcal{E} \rangle_{\mathrm{ex}} \end{array}$$

Question

When can we obtain a heart from a full Ext-exceptional collection? :

$$\begin{array}{ccc} \{\text{heart of finite length}\} & \longrightarrow & \{\text{full Ext-exc. coll.}\} \\ \mathcal{A} & \mapsto & \exists ?\, \mathcal{E} \end{array}$$

such that $\langle \mathcal{E} \rangle_{\mathrm{ex}} \cong \mathcal{A}$.

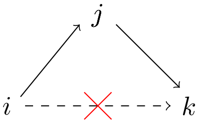
Main results

Let Q be an acyclic quiver and $Q_0 = \{1, \ldots, \mu\}$.

We introduce the two conditions:

(A1) For each $i, j \in Q_0$, we have $\#\{i \to j \in Q_1\} \leq 1$.

(A2) Let $i, j, k = 1, \ldots, \mu$ such that $i < j < k$. If there are arrows from i to j and from j to k, then there are no arrows from i to k.



Examples:

- Dynkin quivers $\vec{\Delta}$.
- Extended Dynkin quivers except for $A^{(1)}_{1,1}$ and $A^{(1)}_{1,2}$.
- Star quivers.

Theorem 7 (O).

Let Q be an acyclic quiver satisfying (A1) (A2), and $\mathcal{A}$ a heart in $\mathcal{D}^b(Q)$. Assume that $\mathcal{A}$ is obtained from $\mathrm{mod}(\mathbb{C}Q)$ by iterated simple tilts. Then, there exists a full Ext-exceptional collection $\mathcal{E} = (E_1, \dots, E_\mu)$ such that $\mathcal{A} = \langle \mathcal{E} \rangle_{\mathrm{ex}}$ and $\mathrm{Sim}\, \mathcal{A} = \{E_1, \dots, E_\mu\}$.

In particular, in the Dynkin cases we have the following

Corollary 8.

Let $\vec{\Delta}$ be a Dynkin quiver.
For each heart $\mathcal{A}$ in $\mathcal{D}^b(\vec{\Delta})$, there exists a full Ext-exceptional collection $\mathcal{E} = (E_1, \dots, E_\mu)$ such that $\mathcal{A} = \langle \mathcal{E} \rangle_{\mathrm{ex}}$ and $\mathrm{Sim}\, \mathcal{A} = \{E_1, \dots, E_\mu\}$.

Sketch of proof of Theorem 7

We prove by induction on the length of simple tiltings.

(Step 1) Case of the standard heart:
The full exceptional collection $\mathcal{E} = (S_1, \ldots, S_\mu)$ consisting of simple modules satisfies the statement.

(Step 2) General case:
By comparing simple tilts with mutations, we show that a full Ext-exceptional collection consisting of simple objects in a given heart $\mathcal{A}$ induces new ones consisting of simple objects in $\mathcal{A}_S^\sharp$ and $\mathcal{A}_S^\flat$.

In order to prove (Step 2), we use the two condition (A1) and (A2). □

Bridgeland stability condition

Let $\mathcal{A}$ be a heart in $\mathcal{D}$ and $K_0(\mathcal{A})$ its Grothendieck group.

Definition 9 (Bridgeland).

1. *Let $\mathcal{A}$ be a heart in $\mathcal{D}$.*
   A stability function $Z\colon K_0(\mathcal{A}) \longrightarrow \mathbb{C}$ is a group homomorphism such that for any nonzero object $E \in \mathcal{A}$ we have

$$Z(E) \in \mathbb{H}_{-} := \{re^{\sqrt{-1}\pi\phi} \in \mathbb{C} \mid R > 0,\ 0 < \phi \leq 1\}.$$

2. *A stability condition on $\mathcal{D}$ is a pair $\sigma = (Z, \mathcal{A})$ consisting of*
   - *a heart $\mathcal{A}$ in $\mathcal{D}$ and,*
   - *a stability function $Z\colon K_0(\mathcal{A}) \longrightarrow \mathbb{C}$ with the "Harder–Narasimhan property" and the "support property".*

Roughly speaking, a stability condition gives a phase and mass for each object and defines σ-(semi)stable objects of $\mathcal{D}$.

Set

$$\mathrm{Stab}(\mathcal{D}) := \{\text{stability condition on } \mathcal{D}\}.$$

It is known that $\exists$ a (generalized) metric on $\mathrm{Stab}(\mathcal{D})$.

Theorem 10 (Bridgeland).

There exists a complex structure on $\mathrm{Stab}(\mathcal{D})$ *of* $\dim_{\mathbb{C}} = \mathrm{rank}_{\mathbb{R}} K_0(\mathcal{D})$.

$\exists$ $\mathbb{C}$-action on $\mathrm{Stab}(\mathcal{D})$:

$$s \cdot (Z, \mathcal{A}) := (Z', \mathcal{A}'), \quad s \in \mathbb{C},\ (Z, \mathcal{A}) \in \mathrm{Stab}(\mathcal{D}),$$

$$\begin{aligned} Z'(E) &:= e^{-\sqrt{-1}\pi s} Z(E), \quad E \in \mathcal{D}, \\ \mathcal{A}' &:= \left\langle E \in \mathcal{D} \;\middle|\; \substack{0 < \phi \leq 1 \\ E \text{ is a semistable object of phase } \phi + \mathrm{Re}(s)} \right\rangle_{\mathrm{ex}} \end{aligned}$$

Comments

- The notion of a stability condition on a triangulated category is a generalization of the slope stability of coherent sheaves on an algebraic curve and King's stability on modules over finite dimensional algebras.
- However, in general, the existence of a stability condition is non-trivial.
- The space of stability conditions $\mathrm{Stab}(\mathcal{D})$ is closely related to several deformation theories (Teichmüller theory, unfolding theory for singularities, A_∞-deformation theory, ...).
- The space of stability conditions $\mathrm{Stab}(\mathcal{D})$ plays an important role in *mirror symmetry*.

Proposition 11.

Let $\mathcal{A}$ be a heart of finite length and $\operatorname{Sim}\mathcal{A} = \{S_1, \dots, S_\mu\}$.
There exists an isomorphism

$$\{(Z, \mathcal{A}) \in \operatorname{Stab}(\mathcal{D})\} \xrightarrow{\cong} \mathbb{H}_-^\mu, \quad (Z, \mathcal{A}) \mapsto (Z(S_1), \dots, Z(S_\mu)).$$

In particular, when $\mathcal{D}$ admits a heart of finite length with finitely many simple objects, $\operatorname{Stab}(\mathcal{D}) \neq \emptyset$.

If a triangulated category $\mathcal{D}$ admits a full exceptional collection, we have the following

Corollary 12 (Macrì).

Let $\mathcal{E} = (E_1, \dots, E_\mu)$ be a full exceptional collection.
There exists a stability conditions σ on $\mathcal{D}$ such that $E_1, \dots, E_\mu$ are σ-stable.

Definition 13 (Dimitrov–Katzarkov).

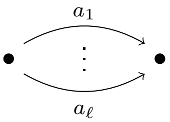
Let $\sigma \in \mathrm{Stab}(\mathcal{D})$ and $\mathcal{E} = (E_1, \ldots, E_\mu)$ an exceptional collection.
$\mathcal{E} = (E_1, \ldots, E_\mu)$ is called σ-exceptional collection if

- *$E_1, \ldots, E_\mu$ are σ-stable,*
- *$\mathcal{E}$ is Ext, and*
- *there exists $r \in \mathbb{R}$ such that $r < \phi(E_i) \leq r + 1$ for all $i = 1, \ldots, \mu$.*

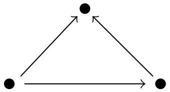
Proposition 14 (Macrì, Dimitrov–Katzarkov).

Let Q be the ℓ-Kronecker quiver K_ℓ or affine $A^{(1)}_{1,2}$ quiver.
For each stability condition σ on $\mathcal{D}^b(Q)$, there exists a full σ-exceptional collection.

K_ℓ :



$A^{(1)}_{1,2}$:



Proposition 15 (O).

Let Q be an acyclic quiver satisfying (A1) (A2), and σ a stability condition on $\mathcal{D}^b(Q)$.
Assume that there exists $s \in \mathbb{C}$ such that the heart $\mathcal{A}'$ of $s \cdot \sigma = (Z', \mathcal{A}')$ is obtained from $\mathrm{mod}(\mathbb{C}Q)$ by iterated simple tilts.
Then, there exists a full σ-exceptional collection.

In the cases of Dynkin quivers, we obtain the next result.

Theorem 16.

Let $\vec{\Delta}$ be a Dynkin quiver.
For each stability condition σ on $\mathcal{D}^b(\vec{\Delta})$, there exists a full σ-exceptional collection.

We expect that the same results hold for the extended Dynkin quivers.

Thank you very much !